

BEHAVIOR OF AN ALMOST SEMICONTINUOUS POISSON PROCESS ON A MARKOV CHAIN UPON ATTAINMENT OF A LEVEL

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We consider the almost semi-continuous processes defined on a finite Markov chain. The representation of the moment generating functions for the absolute maximum after achievement positive level and for the recovery time are obtained. Modified processes with two-step rate of negative jumps are investigated.

1 Introduction

In the present paper, we continue the investigations of almost semicontinuous processes defined on finite Markov chains originated in [1, 2]. In the first section, we consider analogs of the functionals studied in [3] (Sec. 6.3) for the scalar case. In the second section, we study overjump functionals for a modified almost semi-continuous process playing the role of an analog of a modified semicontinuous process with drift whose variations depend on the attained level (see, e.g., [4], Chap. VII and [5]; for the scalar case, see [6]).

Let $x(t)$ be a finite irreducible Markov chain with the set of states $E' = \{1, \dots, m\}$ and an infinitesimal matrix \mathbf{Q} . A process $\xi(t)$ is defined as follows: $\xi(0) = 0$; for $x(t) = k$, $k = 1, \dots, m$, the increments $\xi(t)$ coincide with the increments of the process

$$\xi_k(t) = \sum_{n \leq \varepsilon_k(t)} \xi_n^k - \sum_{n \leq \varepsilon'_k(t)} \xi_n'^k,$$

where $\varepsilon'_k(t)$ and $\varepsilon_k(t)$ are Poisson processes with the rates λ_k^1 and λ_k^2 , respectively. $\xi_n'^k$ and ξ_n^k are independent positive random variables. Moreover, $\xi_n'^k$ have exponential distributions with parameters c_k , whereas ξ_n^k have absolutely continuous distributions with finite expectations m_k . In this case, $Z(t) = \{\xi(t), x(t)\}$ is an almost lower semicontinuous process on a Markov chain (see [1, p. 562]) with cumulant

$$\Psi(\alpha) = \Lambda \mathbf{F}_0(0) \left(\mathbf{C} (\mathbf{C} + i\alpha \mathbf{I})^{-1} - \mathbf{I} \right) + \int_0^\infty (e^{i\alpha x} - \mathbf{I}) \Pi(dx) + \mathbf{Q}, \quad (1)$$

where $\Lambda = \|\delta_{kr}(\lambda_k^1 + \lambda_k^2)\|$, $\mathbf{C} = \|\delta_{kr}c_k\|$, $\mathbf{F}_0(0) = \|\delta_{kr}\lambda_k^1/(\lambda_k^1 + \lambda_k^2)\|$, $\Pi(dx) = \Lambda \bar{\mathbf{F}}_0(0) d\mathbf{F}_0^1(x)$, $\bar{\mathbf{F}}_0(0) = \mathbf{I} - \bar{\mathbf{F}}_0(0)$ and $\mathbf{F}_0^1(x) = \|\delta_{kr} \mathbf{P} \{ \xi_n^k < x \} \parallel$, $x < 0$.

By

$$\xi^\pm(t) = \sup_{0 \leq u \leq t} (\inf) \xi(u), \xi^\pm = \sup_{0 \leq u \leq \infty} (\inf) \xi(u), \bar{\xi}(t) = \xi(t) - \xi^+(t),$$

we denote the extrema of the process $\xi(t)$. The overjump functionals are specified as follows:

$$\tau^+(x) = \inf\{t : \xi(t) > x\}, \gamma^+(x) = \xi(\tau^+(x)) - x, \gamma_+(x) = x - \xi(\tau^+(x) - 0), x \geq 0;$$

$$\tau^-(x) = \inf\{t : \xi(t) < x\}, x \leq 0; \tau^-(x) = 0, x > 0.$$

Let θ_s be an exponentially distributed random variable with parameter $s > 0$ independent of $Z(t)$. The distributions of extrema and the corresponding atomic probabilities are defined as follows:

$$\mathbf{P}_\pm(s, x) = \|\mathbf{P} \{ \xi^\pm(\theta_s) < x, x(\theta_s) = r/x(0) = k \} \| = \mathbf{P} \{ \xi^\pm(\theta_s) < x \}, x > 0;$$

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$$\mathbf{P}^-(s, x) = \mathbf{P} \{ \bar{\xi}(\theta_s) < x \}, \quad x < 0;$$

$\mathbf{p}_\pm(s) = \mathbf{P} \{ \xi^\pm(\theta_s) = 0 \} \mathbf{P}_s^{-1}$, $\mathbf{P}_s = s(s\mathbf{I} - \mathbf{Q})^{-1}$, $\mathbf{p}^-(s) = \mathbf{P}_s^{-1} \mathbf{P} \{ \bar{\xi}(\theta_s) = 0 \}$, and $\mathbf{q}^-(s) = \mathbf{I} - \mathbf{p}^-(s)$. Further, we denote $\mathbf{R}_-(s) = \mathbf{C} \mathbf{p}_-(s)$ and $\mathbf{R}^-(s) = \mathbf{p}^-(s) \mathbf{C}$. Thus, for $x \leq 0$, we can write (see [2])

$$\begin{aligned} \mathbf{P}_-(s, x) &= \mathbf{E} \left[e^{-s\tau^-(x)}, \tau^-(x) < \infty \right] \mathbf{P}_s = \mathbf{q}_-(s) e^{\mathbf{R}_-(s)x} \mathbf{P}_s, \\ \mathbf{P}^-(s, x) &= \mathbf{P}_s e^{\mathbf{R}^-(s)x} \mathbf{q}^-(s). \end{aligned} \quad (2)$$

Since

$$\lim_{x \rightarrow -\infty} \mathbf{P}^-(s, x) = \lim_{x \rightarrow -\infty} \mathbf{P}_-(s, x) = 0,$$

we conclude that the spectra of the matrices $\mathbf{R}_-(s)$ and $\mathbf{R}^-(s)$ ($\sigma(\mathbf{R}_-(s))$ and $\sigma(\mathbf{R}^-(s))$) are formed by positive elements.

The following assertion for overjump functionals is obtained from [2, p. 48]:

Lemma 1. *For the process $Z(t)$ with cumulant (1),*

$$\begin{aligned} \mathbf{f}_s(dx, dy/u) &= \mathbf{E} \left[e^{-s\tau^+(u)}, \gamma_+(u) \in dx, \gamma^+(u) \in dy, \tau^+(u) < \infty \right] = \\ &= s^{-1} d_x \mathbf{P}_+(s, u-x) \mathbf{p}^-(s) \mathbf{\Pi}(dy+x) I\{x < u\} + \\ &\quad + s^{-1} \int_{0 \vee (u-x)}^u d\mathbf{P}_+(s, z) \mathbf{R}^-(s) e^{\mathbf{R}^-(s)(u-x-z)} \mathbf{q}^-(s) \mathbf{\Pi}(dx+y) dy. \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbf{g}_s(dy/u) &= \mathbf{E} \left[e^{-s\tau^+(u)}, \gamma^+(u) \in dy, \tau^+(u) < \infty \right] = s^{-1} \int_0^u d\mathbf{P}_+(s, z) \times \\ &\quad \times \left(\mathbf{p}^-(s) \mathbf{\Pi}(dy+u-z) + \mathbf{R}^-(s) \int_{u-z}^\infty e^{\mathbf{R}^-(s)(u-x-z)} \mathbf{q}^-(s) \mathbf{\Pi}(dx+y) dy \right). \end{aligned} \quad (4)$$

$$\begin{aligned} \mathbf{g}(dy/u) &= \lim_{s \rightarrow 0} \mathbf{g}_s(dy/u) = \\ &= \int_0^u d\mathbf{M}_+(z) \left(\mathbf{\Pi}(dy+u-z) + \mathbf{C} \int_{u-z}^\infty e^{\mathbf{R}^-(0)(u-x-z)} (\mathbf{I} - \mathbf{p}^-(0)) \mathbf{\Pi}(dx+y) dy \right), \end{aligned} \quad (5)$$

where $\mathbf{M}_+(x) = \int_0^\infty e^{i\alpha x} d\mathbf{M}_+(x) = -\Psi^{-1}(\alpha) (\mathbf{C} + i\alpha \mathbf{I})^{-1} (\mathbf{C} \mathbf{p}^-(0) + i\alpha \mathbf{I})$.

Note that $d\mathbf{M}_+(x) = \lim_{s \rightarrow 0} s^{-1} d\mathbf{P}_+(s, x) \mathbf{p}^-(s)$ and the matrices $\mathbf{p}_-(0)$ and $\mathbf{p}^-(0)$ satisfies the equations

$$\begin{aligned} (\mathbf{\Lambda} - \mathbf{Q}) (\mathbf{I} - \mathbf{p}_-(0)) &= \mathbf{\Lambda} \mathbf{F}_0(0) + \int_0^\infty \mathbf{\Pi}(dz) (\mathbf{I} - \mathbf{p}_-(0)) e^{-\mathbf{C} \mathbf{p}_-(0)z}, \\ (\mathbf{I} - \mathbf{p}^-(0)) (\mathbf{\Lambda} - \mathbf{Q}) &= \mathbf{\Lambda} \mathbf{F}_0(0) + \int_0^\infty e^{-\mathbf{p}^-(0) \mathbf{C}z} (\mathbf{I} - \mathbf{p}^-(0)) \mathbf{\Pi}(dz), \end{aligned}$$

respectively.

2 Red period

In the present section, we consider functionals connected with the behavior of $\xi(t)$ upon attainment of a positive level. Denote

$$z^+(u) = \sup_{\tau^+(u) \leq t < \infty} \xi(t) - u, \quad \tau^-(u) = \inf\{t > \tau^+(u), \xi(t) < u\},$$

$$T'(u) = \begin{cases} \tau'(u) - \tau^+(u), & \tau^+(u) < \infty, \\ \infty, & \tau^+(u) = \infty. \end{cases}$$

It is worth noting that the process $Z(t)$ can be regarded as a surplus risk process with stochastic function of premiums (the values of premiums are exponentially distributed) in a Markov environment and the functionals $z^+(u)$, $\tau'(u)$, $T'(u)$ can be regarded as the total deficit after ruin, recovery time, and "red period", respectively (see [7]).

Theorem 1. *For the process $Z(t)$ with cumulant (1)*

$$\mathbf{P} \{z^+(u) < x, \tau^+(u) < \infty\} = \int_0^x \mathbf{g}(dy/u) \mathbf{P} \{\xi^+ < x - y\}. \quad (6)$$

$$\begin{aligned} s\mathbf{E} \left[e^{-s\tau'(u)}, \tau'(u) < \infty \right] &= \int_0^u d\mathbf{P}_+(s, x) \mathbf{p}^-(s) \left(\int_{u-x}^\infty \mathbf{\Pi}(dz) \mathbf{q}_-(s) e^{\mathbf{R}^-(s)(u-x-z)} + \right. \\ &\quad \left. + \mathbf{C} \int_{-\infty}^0 e^{\mathbf{R}^-(s)y} \mathbf{q}_-(s) \int_{u-x-y}^\infty \mathbf{\Pi}(dz) \mathbf{q}_-(s) e^{\mathbf{R}^-(s)(u-x-y-z)} dz \right), \end{aligned} \quad (7)$$

$$\begin{aligned} \mathbf{E} \left[e^{-sT'(u)}, T'(u) < \infty \right] &= \int_0^u d\mathbf{M}_+(x) \left(\int_0^\infty \mathbf{\Pi}(dy + u - x) \mathbf{q}_-(s) e^{-\mathbf{R}^-(s)y} + \right. \\ &\quad \left. + \mathbf{C} \int_{u-x}^\infty e^{\mathbf{R}^-(0)(u-x-z)} \mathbf{q}_-(0) \int_0^\infty \mathbf{\Pi}(dy + z) \mathbf{q}_-(s) e^{-\mathbf{R}^-(s)y} dz \right). \end{aligned} \quad (8)$$

Proof. In view of the fact that, under the condition $\gamma^+(u) \in dy, \tau^+(u) < \infty$, the functional is $z^+(u)$ stochastically equivalent to $y + \xi^+$, we find

$$\mathbf{P} \{z^+(u) < x, \tau^+(u) < \infty\} = \int_0^x \mathbf{P} \{\gamma^+(u) \in dy, \tau^+(u) < \infty\} \mathbf{P} \{y + \xi^+ < x\}.$$

In exactly the same way as in the proof of Theorem 5.1 in [8], for the moment generating function of the time to recovery, we deduce

$$\begin{aligned} \mathbf{E} \left[e^{-s\tau'(u)}, \tau'(u) < \infty \right] &= \int_0^u d\mathbf{P}_+(s, x) \int_{-\infty}^0 \mathbf{P}_s^{-1} \mathbf{P}^-(s, y) \times \\ &\quad \times \int_{u-x-y}^\infty \mathbf{\Pi}(dz) \mathbf{E} \left[e^{-s\tau^-(u-x-y-z)}, \tau^-(u-x-y-z) < \infty \right]. \end{aligned}$$

Combining this relation with (2), we obtain (7). By using the strict Markov property, we get

$$\begin{aligned} &\mathbf{E} \left[e^{-sT'(u)}, T'(u) < \infty, x(T'(u)) = r/x(0) = k \right] = \\ &= \int_0^\infty \mathbf{E} \left[e^{-sT'(u)}, T'(u) < \infty, \gamma^+(u) \in dy, x(T'(u)) = r/x(0) = k \right] = \\ &= \sum_{j=1}^m \int_0^\infty \mathbf{E} \left[e^{-sT'(u)}, T'(u) < \infty, \gamma^+(u) \in dy, x(T'(u)) = r, x(\tau^+(u)) = j/x(0) = k \right] \\ &= \sum_{j=1}^m \int_0^\infty \mathbf{P} \{x(\tau^+(u)) = j, \gamma^+(u) \in dy, \tau^+(u) < \infty/x(0) = k\} \times \\ &\quad \times \mathbf{E} \left[e^{-sT'(u)}, T'(u) < \infty, x(T'(u)) = r/\gamma^+(u) \in dy, x(\tau^+(u)) = j, \tau^+(u) < \infty, x(0) = k \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^m \int_0^\infty \mathbf{E} \left[e^{-s\tau^-(-y)}, \tau^-(-y) < \infty, x(\tau^-(-y)) = r/x(0) = j \right] \times \\
&\quad \times \mathbf{P} \{ x(\tau^+(u)) = j, \gamma^+(x) \in dy, \tau^+(u) < \infty/x(0) = k \}.
\end{aligned}$$

In deducing this equality, we have used the fact that, under the condition $\gamma^+(u) \in dy, \tau^+(u) < \infty$, the functional $T'(u)$ is stochastically equivalent to the time of attainment of the level $-y$. In the matrix form, we can write

$$\begin{aligned}
\mathbf{E} \left[e^{-sT'(u)}, T'(u) < \infty, \tau^+(u) < \infty \right] &= \\
&= \int_0^\infty \mathbf{P} \{ \gamma^+(u) \in dy, \tau^+(u) < \infty \} \mathbf{E} \left[e^{-s\tau^-(-y)}, \tau^-(-y) < \infty \right].
\end{aligned}$$

By using (2) and (5), we establish equality (8). \square

3 Modified Process

In the present section, in addition to the results obtained for the overjump functionals, we use the relations for two-limit functionals. By

$$\tau(u, b) = \{t > 0 : \xi(t) \notin (u - b, u)\}$$

we denote the time of exit from the interval $(u - b, u)$. Further, we consider the events specifying the times of exit through the upper and lower boundaries of the interval:

$$A_+(u) = \{\omega : \xi(\tau(u, b)) \geq u\} \text{ and } A_-(u) = \{\omega : \xi(\tau(u, b)) \leq u - b\},$$

and the corresponding overjumps

$$\begin{aligned}
\gamma_b^+(u) &= \xi(\tau(u, b)) - u, \gamma_+^b(u) = u - \xi(\tau(u, b) - 0) \text{ on } A_+(u); \\
\gamma_b^-(u) &= (u - b) - \xi(\tau(u, b)), \gamma_-^b(u) = \xi(\tau(u, b) - 0) - (u - b) \text{ on } A_-(u).
\end{aligned}$$

It follows from the results presented in [1, p.559] that

$$\mathbf{E} \left[e^{-s\tau(u, b)}, \gamma_b^-(u) \in dy, A_-(u) \right] = \mathbf{E} \left[e^{-s\tau(u, b)}, A_-(u) \right] \mathbf{C} e^{-\mathbf{C}y} dy = \mathbf{B}_b(s, u) \mathbf{C} e^{-\mathbf{C}y} dy, \quad (9)$$

$$\begin{aligned}
\mathbf{f}_{b,s}^+(dx, dy/u) &= \mathbf{E} \left[e^{-s\tau(u, b)}, \gamma_+^b(u) \in dx, \gamma_b^+(u) \in dy, A_+(u) \right] = \\
&= \left\| \mathbf{P} \{ u - \xi(\theta_s) \in dx, \tau(u, b) > \theta_s, x(\theta_s) = r/x(0) = k \} \right\| \mathbf{\Pi}(dy + x) I\{0 < x < b\} = \\
&= d_x \mathbf{H}_s(b, u, u - x) \mathbf{\Pi}(dy + x) I\{0 < x < b\}.
\end{aligned}$$

Note that the representations for $\mathbf{B}_b(s, u)$ and $d_x \mathbf{H}_s(b, u, x)$ were obtained in [1] for almost upper semicontinuous processes. In this case, one can use the fact that if $\{\xi(t), x(t)\}$ is an almost upper semicontinuous process, then $\{-\xi(t), x(t)\}$ is an almost lower semicontinuous process.

We now determine the modified process $\xi_{a,b}(t)$, $0 < a \leq b < \infty$. Assume that the rates of exponentially distributed negative jumps $\xi_{a,b}(t)$ depend on the threshold levels a and b (see [6]). In the risk theory, this process has the following interpretation: As soon as the reserve of an insurance company attains a certain level, the company may decrease the value of the premium to attract additional clients. Therefore, the distribution of the values of premiums contains a parameter $\tilde{\mathbf{C}} = \mathbf{C}(r)$, if the reserve of the company is equal to r . Assume that $\tilde{\mathbf{C}}$ takes only two values \mathbf{C} and \mathbf{C}_* equal to the initial and

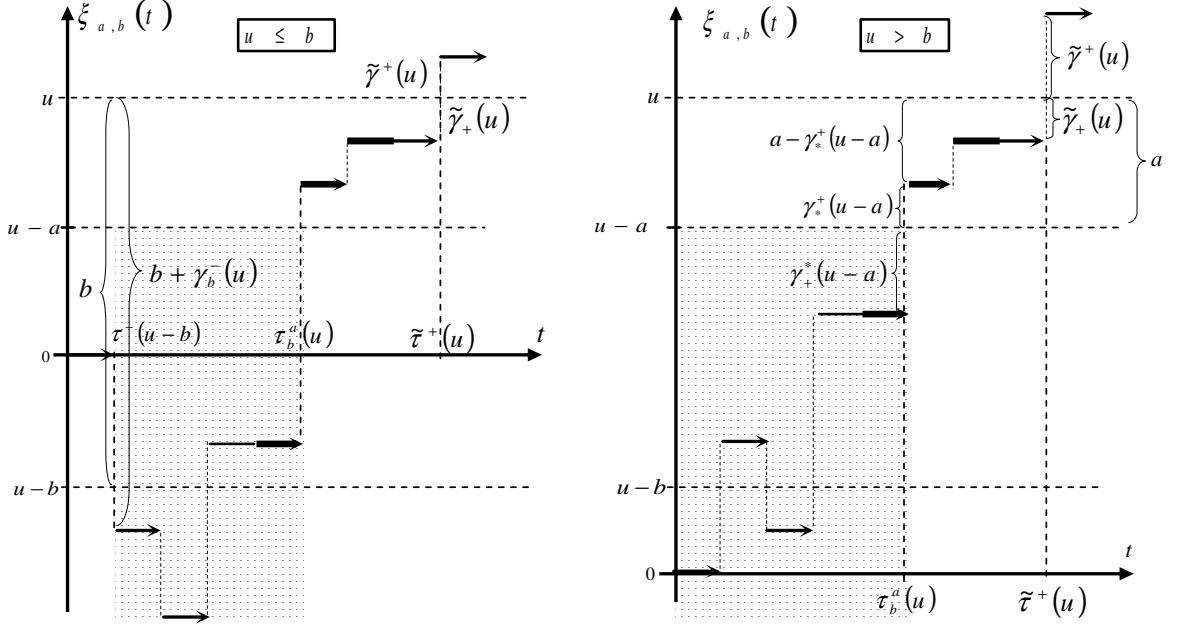


Figure 1: Modified Risk Process

lowered values of the premium, respectively, and in addition, that the transition between these values occurs on passing through an inert zone (a, b) .

The increments of the process $\xi_{a,b}(t)$ coincide with the increments of the process $\xi(t)$ (with intensities \mathbf{C}) between the last crossing of the level $u - a$ from below and the next crossing of the level $u - b$ from above. The increments of $\xi_{a,b}(t)$ coincide with $\xi_*(t)$ (with intensities \mathbf{C}_*) between the last crossing of the level $u - b$ from above and the subsequent crossing of the level $u - a$ from below. In the notation of moment generating functions we use the symbol $*$, which corresponds to the process $\xi_*(t)$. Further, if $x(t) = k$, then

$$d\xi_{a,b}(t) = d\xi_k(t)I\{0 \leq t \leq \tau^-(u-b)\} + d\xi_k^*(t)I\{\tau^-(u-b) < t \leq \tau_b^a(u)\} + d\xi_{a,b}(t - \tau_b^a(u))I\{t > \tau_b^a(u)\},$$

where $\tau_b^a(u) = \inf\{t > \tau^-(u-b) : \xi_{a,b}(t) \geq u-a\}$.

By $\tilde{\tau}^+(u), \tilde{\gamma}_+(u), \tilde{\gamma}^+(u)$ we denote the overjump functionals for the modified process $\xi_{a,b}(t)$ (see Figure 1). We also denote

$$\mathbf{f}_s^{a,b}(dx, dy/u) = \mathbf{E} \left[e^{-s\tilde{\tau}^+(u)}, \tilde{\gamma}_+(u) \in dx, \tilde{\gamma}^+(u) \in dy, \tilde{\tau}^+(u) < \infty \right].$$

Then the Gerber-Shiu function can be defined as follows (see [9])

$$\Phi_s^{a,b}(u) = \int_0^\infty \int_0^\infty w(x, y) \mathbf{f}_s^{a,b}(dx, dy/u),$$

where $w(x, y)$, $x, y > 0$ is a nonnegative function (penalty). If the parameter s is regarded as the force of interest, then $\Phi_s^{a,b}(u)$ can be regarded as a discounted expected penalty at the time to ruin.

Assume that the process $e^{u-\xi_{a,b}(t)}$ describes the price of a stock whose variations have the form of random jumps. We now consider a perpetual American put option with strike price K . The payoff at time t is equal to $(K - e^{u-\xi_{a,b}(t)})_+$. For the scalar case, the optimal strategy is as follows

$$\tau_\beta = \inf\{t > 0 : e^{u-\xi_{a,b}(t)} < e^\beta\},$$

where exercise boundary β : $e^\beta \leq \min(e^u, K)$. Assume that the market is risk neutral. Then the price of an option is defined as the expected discounted payoff

$$\mathbf{E} \left[e^{-s\tau_\beta} (K - e^{u - \xi_{a,b}(\tau_\beta)})_+ \right]$$

or, in view of the fact that $\tau_\beta \doteq \tilde{\tau}^+(u - \beta)$, as follows

$$\mathbf{E} \left[e^{-s\tilde{\tau}^+(u-\beta)} (K - e^{\beta - \tilde{\gamma}^+(u-\beta)})_+ \right].$$

Therefore, $\Phi_s^{a,b}(u - \beta)$ with $w(x, y) = (K - e^{\beta - y})_+$ can also be regarded as the price of perpetual American put option [10, p.12].

Theorem 2. For the modified process $\{\xi_{a,b}(t), x(t)\}$

1) if $0 < u \leq b$, then

$$\mathbf{f}_s^{a,b}(dx, dy/u) = \mathbf{f}_{b,s}^+(dx, dy/u) + \mathbf{B}_b(s, u) \int_0^\infty \mathbf{C}e^{-\mathbf{C}z} \mathbf{f}_s^{a,b}(dx, dy/z + b) dz; \quad (10)$$

2) if $b < u$, then

$$\begin{aligned} \mathbf{f}_s^{a,b}(dx, dy/u) &= \mathbf{f}_s^*(dx - a, dy + a/u - a) I\{x > a\} + \\ &+ \int_0^a \mathbf{g}_s^*(dz/u - a) \left(\mathbf{f}_{b,s}^+(dx, dy/a - z) + \mathbf{B}_b(s, a - z) \int_0^\infty \mathbf{C}e^{-\mathbf{C}v} \mathbf{f}_s^{a,b}(dx, dy/v + b) dv \right), \end{aligned} \quad (11)$$

where

$$\begin{aligned} &\left(\mathbf{I} - \int_0^\infty \mathbf{C}e^{-\mathbf{C}z} \int_0^a \mathbf{g}_s^*(dv/b - a + z) \mathbf{B}_b(s, a - v) dz \right) \int_0^\infty \mathbf{C}e^{-\mathbf{C}z} \mathbf{f}_s^{a,b}(dx, dy/z + b) dz = \\ &= \int_0^\infty \mathbf{C}e^{-\mathbf{C}z} \left(\mathbf{f}_s^*(dx - a, dy + a/b - a + z) I\{x > a\} + \int_0^a \mathbf{g}_s^*(dv/b - a + z) \mathbf{f}_{b,s}^+(dx, dy/a - v) \right) dz. \end{aligned} \quad (12)$$

Proof. Relation (10) is an analog of the result obtained in [6]. By using the formula of total probability and the strict Markov property, for $u > b$, we find

$$\begin{aligned} &\mathbf{E} \left[e^{-s\tilde{\tau}^+(u)} , \tilde{\gamma}_+(u) \in dx, \tilde{\gamma}^+(u) \in dy, \tilde{\tau}^+(u) < \infty \right] = \\ &= \mathbf{E} \left[e^{-s\tau_*^+(u-a)} , \gamma_+^*(u-a) + a \in dx, \gamma_*^+(u-a) - a \in dy, \gamma_*^+(u-a) > a, \tau_*^+(u-a) < \infty \right] + \\ &+ \int_0^a \mathbf{E} \left[e^{-s\tau_*^+(u-a)} , \gamma_*^+(u-a) \in dz, \tau_*^+(u-a) < \infty \right] \times \\ &\times \mathbf{E} \left[e^{-s\tilde{\tau}^+(a-z)} , \tilde{\gamma}_+(a-z) \in dx, \tilde{\gamma}^+(a-z) \in dy, \tilde{\tau}^+(a-z) < \infty \right]. \end{aligned} \quad (13)$$

This yields relation (11). Relation (12) is obtained from (11) as a result of the integral transform. \square

In the scalar case ($m = 1$) the matrix relations become somewhat simpler. If we set $w(x, y) = 1$, then $\Phi_0^{a,b}(u)$ is the ruin probability for the modified process.

Corollary 1. For the scalar modified process $\xi_{a,b}(t)$

1) if $0 < u \leq b$, then

$$\Phi_0^{a,b}(u) = 1 - B_b(u) \left(1 - \int_0^a g_*(dz/b - a + \theta'_c) B_b(a - z) \right)^{-1} P_+^*(b - a + \theta'_c); \quad (14)$$

2) if $b < u$ then

$$\begin{aligned} \Phi_0^{a,b}(u) = P_+^*(u-a) - \int_0^a g_*(dz/u-a) B_b(a-z) \times \\ \times \left(1 - \int_0^a g_*(dz/b-a+\theta'_c) B_b(a-z)\right)^{-1} P_+^*(b-a+\theta'_c), \end{aligned} \quad (15)$$

where

$$\begin{aligned} P_+^*(b-a+\theta'_c) &= \int_0^\infty ce^{-cx} P\{\xi_*^+ < b-a+x\} dx, \\ g_*(dz/b-a+\theta'_c) &= \int_0^\infty ce^{-cx} P\{\gamma_*^+(b-a+x) \in dz, \text{ and } \tau_*^+(b-a+x) < \infty\} dx. \end{aligned}$$

Example. Assume that, for the scalar risk process, the premiums ξ'_n have exponential distributions with parameter \tilde{c} , whereas the claims ξ_n obey the Erlang distribution(2):

$$P\{\xi_n < x\} = \delta^2 x e^{-\delta x}, x > 0.$$

It is necessary to determine the corresponding ruin probability for $u \leq b$.

According to Example 5.2 [3], for $E\xi(1) < 0$ and $E\xi_*(1) < 0$ we find

$$\begin{aligned} P\{\xi_*^+ < u\} &= P_+^*(u) = 1 - a_1^* e^{-r_1^* u} - a_2^* e^{-r_2^* u}; \\ M_+^*(0+) &= \frac{1}{\lambda}, dM_+^*(x) = \frac{1}{c_* |E\xi_*(1)|} dP_+^*(x), R^-(0) = p^-(0) = 0, \\ B_b(u) &= (1 - a_1 e^{-r_1 u} - a_2 e^{-r_2 u}) (1 - b_1 e^{-r_1 b} - b_2 e^{-r_2 b})^{-1}, \end{aligned}$$

where the quantities a_i, a_i^* , and b_i are independent of u and b ; r_i and r_i^* are positive roots of the Lundberg equation for the processes $\xi(t)$ and $\xi_*(t)$, respectively. By using (5) and (14), we conclude (for $u \leq b$):

$$P\{\tilde{\tau}^+(u) < \infty\} = 1 - \frac{(1 - a_1 e^{-r_1 u} - a_2 e^{-r_2 u})(1 - f_1^* e^{-r_1^*(b-a)} - f_2^* e^{-r_2^*(b-a)})}{P(u, a, b)}, \quad (16)$$

$$\begin{aligned} P(u, a, b) &= 1 - f_2 e^{-r_1 b} - f_3 e^{-r_2 b} + (g_{11} + g_{12}(b-a))e^{-\delta(b-a)} + \\ &+ (g_{21} + g_{22}(b-a))e^{-\delta(b-a)-r_1 a} + (g_{31} + g_{32}(b-a))e^{-\delta(b-a)-r_2 a}, \end{aligned}$$

where f_i, f_i^* , and $g_{i,j}$ are independent of u, a and b .

Assume that $c = 1$, $c_* = 4$, $\delta = 20$, $\lambda_1 = 2$, and $\lambda_2 = 1$. Then $E\xi(1) = -19/10$ and $E\xi_*(1) = -2/5$. The corresponding Lundberg roots are $r_1 = 8$, $r_2 = 95/3$ and $r_1^* = 20/3$, $r_2^* = 32$.

In addition, the distribution of the absolute maximum of the process $\xi_*(t)$ and the probability of exit of the process $\xi(t)$ from the interval $(u-b, b)$ through the lower boundary are given by the formulas

$$P\{\xi_*^+ < u\} = 1 - \left(\frac{32}{57} e^{-20u/3} - \frac{9}{95} e^{-32u}\right);$$

$$B_b(u) = P\{\xi(\tau(u, b)) \leq u-b\} = \frac{1 + \frac{49}{426} e^{-95u/3} - \frac{171}{355} e^{-8u}}{1 + \frac{1}{284} e^{-95b/3} - \frac{19}{355} e^{-8b}}.$$

For $a = b$, by using (16), we arrive at the following expression for the ruin probability

$$P\{\tilde{\tau}^+(u) < \infty\} = 1 - \frac{1 + \frac{49}{426} e^{-95u/3} - \frac{171}{355} e^{-8u}}{1 - \frac{45}{111328} e^{-95b/3} + \frac{19}{852} e^{-8b}}.$$

References

- [1] Karnaukh, E.V. Two-limit problems for almost semicontinuous processes defined on a Markov chain. (Ukrainian, English)//Ukr. Mat. Zh. 59, No. 4 (2007), 555-565, translation in Ukr. Math. J. 59, No. 4 (2007), 620-632, arXiv:0909.1420v1[math.PR].
- [2] Karnaukh, E.V. Overshoot functionals for almost semicontinuous processes defined on a Markov chain // Teor. Imovir. ta Matem. Statyst., No.76 (2007), 45-52, translation in Theor. Probability and Math. Statist. No. 76 (2008), 49-57, arXiv:0909.3690v1[mathPR].
- [3] Gusak D.V., *Boundary value problems for processes with independent increments in the risk theory*[in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, 2007.
- [4] Asmussen S., *Ruin Probabilities*, World Scientific, Singapore, 2000.
- [5] Jasiulewicz H., *Probability of ruin with variable premium rate in a Markovian environment*, Insurance: Mathematics and Economics **29** (2001), 291–296.
- [6] Bratiychuk M.S., Derfla D., *On a modification of the classical risk process*, Insurance: Mathematics and Economics **41** (2007), 156–162.
- [7] Rolsky T., Shmidly H., Shmidt V., Teugels J., *Stochastic Processes for Insurance and Finance*, Wiley, New York, 1999.
- [8] Gusak D.V., *Boundary problems for processes with independent increments on Markov chains and for semi-Markov processes*, Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, 1998.
- [9] Gerber H.U., Shiu S.W., *On the time value of ruin*, North American Actuarial Journal **2** (1998), N 1, 48–78.
- [10] Gerber H.U., Shiu S.W., *From ruin theory to pricing reset guarantees and perpetual put options*, Insurance: Mathematics and Economics **24** (1999), 3–14.